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ASYMPTOTIC UNBOUNDED ROOT LOCI - FORMULAE AND COMPUTATION

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ABSTRACT

We study the asymptotic behavior of the closed loop eigenvalues (root loci) of a strictly proper, linear, time-invariant control system as loop gain goes to  $\infty$ . The formulae are stated in terms of the eigenvalues of nested restricted linear maps of the form  $A(\text{mod } S_2) \big|_{S_1}$  where  $S_1$  and  $S_2$  are subspaces of complementary dimension. Additional geometrical insight into the formulae is obtained by mechanizing the formulae using orthogonal projections.

Our method and formulae are useful in other asymptotic calculations as well e.g. hierarchical multiple-time scales aggregation of Markov chains with some infrequent transitions.

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## Section 1. Introduction

High loop gain enhances the desirable effects of feedback e.g. desensitization and disturbance attenuation. However, most practical control systems are driven to instability by excessively high gain feedback. With this design tradeoff in mind, this paper presents a new geometric way of computing the asymptotic behavior of unbounded root loci of a strictly proper, linear, time invariant control system shown in Figure 1 as loop gain  $\rightarrow \infty (k \rightarrow \infty)$ .

The asymptotic behavior has been extensively studied by Kouvaritakis and Shaked [3], Kouvaritakis [4], Kouvaritakis and Edmunds [5], Owens [11,12], MacFarlane and co-workers [2,19], Kokotovic, et. al [22], and by Brockett and Byrnes [10]. While it is easy to check that our formulae agree with those of the above authors, our contribution we feel is threefold:

(i) In the process of developing geometric formulae for the asymptotes of the unbounded root loci we have defined (motivated by Wonham [18]) and developed some properties of restricted linear maps of the form  $A(\text{mod } S_2) \Big|_{S_1}$ , where  $A$  is a map from  $\mathbb{C}^n$  to  $\mathbb{C}^n$  and  $S_1, S_2$  are subspaces of complementary dimension. These restricted maps and nested versions thereof are also of independent interest.

(ii) We have established under a certain simple null structure assumption the exact connection in our setting between the structure at  $\infty$  of the linear system (see e.g. Verghese, et. al. [20], van Dooren, et. al. [7]) and the unbounded asymptotic root loci.

(iii) Our techniques of asymptotic analysis are useful in other problems as well: We have applied in [21] exactly the same asymptotic formulae to the study of the hierarchical aggregation of linear systems and finite state Markov processes with multiple time scales - an example of singular perturbation with several time scales. Also the relevance of these calculations to the asymptotic behavior of bounded root loci is explicated in the Conclusions (section 6).

The organization of the paper is as follows: In section 2, we develop properties of restricted linear maps of the form  $A \pmod{S_2} \Big|_{S_1}$  and nested restrictions of linear maps. In section 3, we obtain formulae for the leading coefficient of the asymptotic values of the unbounded multivariable root loci in terms of eigenvalues of certain maps of the form studied in section 2. Using some results of van Dooren, et.al. [7] and Verghese et.al. [20] and a certain simple null structure assumption we relate these asymptotic values to the structure at  $\infty$  of the Smith McMillan form of the open loop transfer function. In section 4, we give explicit matrix formulae for the (more abstract) formulae of section 3. By the artifice of using orthogonal projections and the singular value decomposition in this development, we develop additional geometric insight - identify subspaces of the input space and output space where effects of the  $O(k)$ ,  $O(k^{1/2})$ ,  $O(k^{1/3})$ , ... unbounded root loci dominate asymptotically (the  $O$  and  $o$  notation used in this paper are standard, see e.g. [6]).

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We calculate formulae for the pivots of the unbounded root loci and show them to have the same form as the coefficients of the unbounded asymptotic root loci, in section 5. Concluding remarks on relaxing the assumptions of our work are collected in section 6.

## Notation

(i)  $R(A)$  stands for the range of a matrix  $A \in \mathbb{C}^{n \times m}$  and

$N(A)$  stands for the nullspace of  $A$ .

(ii)  $A^\dagger$  stands for any right (pseudo) inverse of  $A$ , defined (non-uniquely) as follows:

Let  $f_1, \dots, f_k$  be a basis for the  $R(A)$  with  $e_1, \dots, e_k$  chosen such that  $Ae_i = f_i$ ,  $i = 1, \dots, k$ .

Complete the basis  $f_1, \dots, f_k$  to obtain a basis  $f_1, \dots, f_n$  of  $\mathbb{C}^n$ . Now define

$$\begin{aligned} A^\dagger f_i &= e_i & i &= 1, \dots, k \\ A^\dagger f_i &= 0 & i &= k+1, \dots, n \end{aligned}$$

## Section 2. Restrictions of a Linear Map

### 2.1 General Theory

Given a linear map  $A$  from  $\mathbb{C}^n$  to  $\mathbb{C}^n$  and a subspace  $S_2 \subset \mathbb{C}^n$  of dimension  $(n-m)$ , the operator  $A(\text{mod } S_2)$  is the linear map from  $\mathbb{C}^n$  to  $\mathbb{C}^n/S_2$  defined by the following diagram:

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{A} & \mathbb{C}^n \\ & \searrow & \downarrow P \\ (A(\text{mod } S_2) = P \cdot A) & & \mathbb{C}^n/S_2 \end{array}$$

Here,  $P$  stands for the canonical projection  $\mathbb{C}^n \rightarrow \mathbb{C}^n/S_2$ . The maps whose structure we will explore here are of the form  $A(\text{mod } S_2)|_{S_1}$  where  $S_1 \subset \mathbb{C}^n$  is a subspace of dimension complimentary to  $S_2$  (namely,  $m$ ). Pictorially, we have for

$$A(\text{mod } S_2) \Big|_{S_1}$$

$$A(\text{mod } S_2) \Big|_{S_1} = P \cdot A \cdot i$$

$$\begin{array}{ccccc} S_1 & \xrightarrow{i} & \mathbb{C}^n & \xrightarrow{A} & \mathbb{C}^n \\ & & & & \downarrow P \\ & & & & \mathbb{C}^n / S_2 \end{array}$$

Here  $i$  is the (canonical) inclusion map of  $S_1$  in  $\mathbb{C}^n$ . Note that if  $\hat{S}_2$  is any direct summand of  $S_2$  then  $\hat{S}_2$  is isomorphic to  $\mathbb{C}^n / S_2$ . Since  $\mathbb{C}^n / S_2$  is an abstract vector space, we will for the purpose of computation identify  $\mathbb{C}^n / S_2$  with  $\hat{S}_2$ . We have then the following representation theorem for  $A(\text{mod } S_2) \Big|_{S_1}$ .

Theorem 2.1 (Representation Theorem)

Let the columns of  $T_1 \in \mathbb{C}^{n \times m}$  form a basis for  $S_1$ , and the columns of  $T_2 \in \mathbb{C}^{n \times m}$  form a basis for  $\hat{S}_2$ , some direct summand of  $S_2$ . Then, the matrix representation for  $A(\text{mod } S_2) \Big|_{S_1}$  with respect to the bases furnished by  $T_1$  and  $T_2$  is

$$(T_2^* T_2)^{-1} T_2^* A T_1 \in \mathbb{C}^{m \times m}. \quad (2.1)$$

Proof: Recall from elementary linear algebra [1, pg. 125] that

$T_2(T_2^* T_2)^{-1} T_2^* \in \mathbb{C}^{n \times n}$  is the matrix representation of the projection from  $\mathbb{C}^n$  onto  $\hat{S}_2$  with the columns of  $T_2$  as basis for  $\hat{S}_2$ . Since the columns of  $T_1$ ,  $T_2$  are chosen as bases for  $S_1$ ,  $\hat{S}_2$  the result follows.  $\square$

Notes: (i)  $(T_2^* T_2)^{-1} T_2^*$  is a left inverse of  $T_2$ .

(ii) If the columns of  $\tilde{T}_1 \in \mathbb{C}^{n \times m}$  and  $\tilde{T}_2 \in \mathbb{C}^{n \times m}$  form bases for  $S_1$  and  $S_2$  (any other direct summand of  $S_2$ ), then the representations are related by  $(\tilde{T}_2^* \tilde{T}_2)^{-1} \tilde{T}_2^* A \tilde{T}_1 = P(T_2^* T_2)^{-1} T_2^* A T_1 Q$ , where  $P, Q \in \mathbb{C}^{m \times m}$  are nonsingular matrices.

Definition 2.2  $\lambda \in \mathbb{C}$  is an eigenvalue of  $A(\text{mod } S_2)|_{S_1}$  if  $\exists$  non-zero  $x \in S_1$  such that  $(A - \lambda I)x \text{ (mod } S_2) = 0$ ; equivalently,  $\exists x \in S_1 \ni (A - \lambda I)x \in S_2$ .

Proposition 2.3 (Generalized eigenvalue problem for eigenvalues of  $A(\text{mod } S_2)|_{S_1}$ ).

Let  $B \in \mathbb{C}^{n \times m}$ ;  $C \in \mathbb{C}^{m \times n}$  be chosen so that  $R(B) = S_2$ ,  $\text{Ker } C = S_1$ . Then, the eigenvalues of  $A(\text{mod } S_2)|_{S_1}$  are precisely the solutions,  $\lambda$ , of the generalized eigenvalue problem

$$\begin{bmatrix} A - \lambda I & | & B \\ \hline & & \\ & & \\ & C & | & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0 \quad (2.2)$$

with  $x \neq 0 \in \mathbb{C}^n$  and  $u \in \mathbb{C}^m$

Proof: Rewriting (2.2), we have with  $x \neq 0$

$$(A - \lambda I)x + Bu = 0$$

with

$$x \in \text{Ker } C. \quad \square$$

In fact solutions of all generalized eigenvalue problems can be obtained from Definition 2.1 as follows:

Proposition 2.4 (converse to Proposition 2.3)

The solutions  $\lambda$  of the generalized eigenvalue problem

$$\begin{bmatrix} A - \lambda I & | & B \\ \hline & & \\ & & \\ & C & | & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0 \quad (2.3)$$

with  $x \neq 0 \in \mathbb{C}^n$  and  $u \in \mathbb{C}^m$  are the eigenvalues of

$$A - BD^\dagger C \pmod{B(N(D))} \Big|_{C^{-1}(R(D))} \quad (2.4)$$

(Here,  $D^\dagger$  stands for any right inverse of  $D$ ,  $C^{-1}(R(D))$  stands for the inverse image under  $C$  of  $R(D)$  and  $B(N(D))$  stands for the image under  $B$  of  $N(D)$ ).

Proof: Rewriting (2.3) we have with  $x \neq 0$

$$(A - \lambda I)x + Bu = 0 \quad (2.5)$$

$$Cx + Du = 0 \quad (2.6)$$

Note that  $Cx$  must belong to the range of  $D$  so that  $x \in C^{-1}(R(D))$  and the (non-unique) solution of (2.6) is

$$u = -D^\dagger Cx + v \quad (2.7)$$

where  $v$  is any element of  $N(D)$

Use (2.7) in (2.5) to obtain

$$(A - BD^\dagger C - \lambda I)x + Bv = 0$$

with  $x \in C^{-1}(R(D))$  and  $v \in N(D)$ .

The converse is similar. □

The concept of a right pseudo-inverse may be generalized to operators of the form  $A \pmod{S_2} \Big|_{S_1}$ .

Definition 2.5 A right pseudo-inverse of  $A \pmod{S_2} \Big|_{S_1}$  denoted  $(A \pmod{S_2} \Big|_{S_1})^\dagger$  is defined from  $\mathbb{C}^n/S_2$  to  $S_1$  as follows:

Let  $f_1, \dots, f_p$  be a basis for the range of  $A \pmod{S_2} \Big|_{S_1}$  in  $\mathbb{C}^n/S_2$ .

Choose  $e_1, \dots, e_p$  belonging to  $S_1$  such that

$$A \pmod{S_2} e_i = f_i \quad i = 1, \dots, p.$$

Complete the basis  $f_1, \dots, f_p$  to obtain a basis  $f_1, \dots, f_m$  of  $\mathbb{C}^n/S_2$ . Now

define

$$\begin{aligned} (A(\text{mod } S_2) \Big|_{S_1})^\dagger f_i &= e_i & i = 1, \dots, p \\ (A(\text{mod } S_2) \Big|_{S_1})^\dagger f_i &= 0 & i = p+1, \dots, m \end{aligned}$$

The setting in which the definitions made above are most useful is when  $S_1 \oplus S_2 = \mathbb{C}^n$ . We specialize to this case now. There is then a natural isomorphism  $\tilde{I}$  between  $S_1$  and  $\mathbb{C}^n/S_2$  as follows

$$\begin{array}{ccccc} S_1 & \xrightarrow{i} & \mathbb{C}^n & \xrightarrow{I} & \mathbb{C}^n \\ & \searrow & & \downarrow p & \\ & & & \mathbb{C}^n/S_2 & \end{array} \quad \tilde{I} := I(\text{mod } S_2) \Big|_{S_1}$$

$\tilde{I} = p \cdot I \cdot i$

In this case  $S_1$  and  $\mathbb{C}^n/S_2$  are abstractly the same space so that the terminology eigenvalue of  $A(\text{mod } S_2) \Big|_{S_1}$  is intuitive. It is clear that

in this case, there are  $m$  eigenvalues of  $A(\text{mod } S_2) \Big|_{S_1}$ . We explore their structure.

Definition 2.6  $A(\text{mod } S_2) \Big|_{S_1}$  is said to have simple null structure if

there does not exist  $x \in S_1$  such that

$$A(\text{mod } S_2)x \neq 0 \quad \text{and} \quad A(\text{mod } S_2) \tilde{I}^{-1} A(\text{mod } S_2)x = 0.$$

Comments: (i) The definition states that there are no generalized eigenvectors associated with the eigenvalue  $\lambda = 0$  of

$$A(\text{mod } S_2) \Big|_{S_1}.$$



- (ii) Definition 2.6 is useful for counting the number of non-zero (possibly repeated) eigenvalues of  $A(\text{mod } S_2) \Big|_{S_1}$  as follows:

Proposition 2.7 (Number of non zero eigenvalues of  $A(\text{mod } S_2) \Big|_{S_1}$ ).

If  $A(\text{mod } S_2) \Big|_{S_1}$  has simple null structure the number (counting multiplicities) of its non-zero eigenvalues is equal to its rank, namely the dimension of  $R(A(\text{mod } S_2) \Big|_{S_1})$ .

Definition 2.8 :  $A(\text{mod } S_2) \Big|_{S_1}$  is said to have simple structure associated with an eigenvalue  $\lambda$  if  $(A - \lambda I)(\text{mod } S_2) \Big|_{S_1}$  has simple null structure.

With these definitions on hand, one may state the Jordan canonical form theorem for the operator  $A(\text{mod } S_2) \Big|_{S_1}$ .

Theorem 2.9 (Jordan Canonical form for  $A(\text{mod } S_2) \Big|_{S_1}$ )

Assume  $S_1 \oplus S_2 = \mathbb{C}^n$  and identify  $\mathbb{C}^n / S_2$  with  $S_1$ . Then, there exists a choice of basis for  $S_1$  - the columns of  $T \in \mathbb{C}^{n \times m}$  such that the matrix representation of  $A(\text{mod } S_2) \Big|_{S_1} : S_1 \rightarrow \mathbb{C}^n / S_2 \cong S_1$  is

$$(T^* T)^{-1} T^* A T = \text{diag } [J_1, \dots, J_p] \quad (2.8)$$

where  $J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & \\ & & & \lambda_i \end{bmatrix}$ .

Since  $(T^* T)^{-1} T^* = T^L$ , a left inverse of  $T$ , we may write (2.8) as

$$T^L A T = \text{diag } [J_1, \dots, J_p] \quad .$$

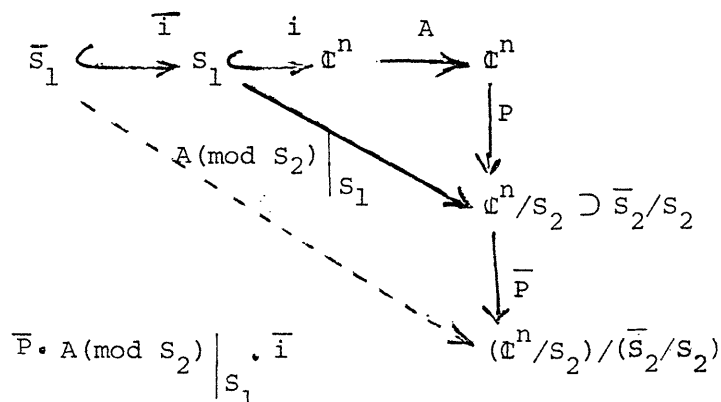
Proof: Follows exactly along the same lines as the regular Jordan canonical form theorem and is omitted.

□

## 2.2 Nested Restrictions of a Linear Map

Given subspaces  $\bar{S}_1 \subset S_1$ ,  $\bar{S}_2, S_2$  subspaces of  $\mathbb{T}^n$ ,  $A(\text{mod } S_2) \Big|_{S_1} \text{ mod } (\bar{S}_2/S_2) \Big|_{\bar{S}_1}$

is defined naturally by the dotted arrow in the following commuting diagram



The cumbersome notation  $A(\text{mod } S_2) \Big|_{S_1} (\text{mod } \bar{S}_2/S_2) \Big|_{\bar{S}_1}$  is replaced by

$$A(\text{mod } S_2) (\text{mod } \bar{S}_2/S_2) \Big|_{\bar{S}_1}.$$

Matrix representation for these operators is similar in spirit to Theorem 2.1: First, identify  $\mathbb{T}^n/S_2$  with any direct summand of  $S_2$  in  $\mathbb{T}^n$ ; say  $\hat{S}_2$ . Let the columns of  $T_1, T_2 \in \mathbb{T}^{n \times m}$  span  $S_1, \hat{S}_2$  respectively. With these bases for  $S_1$  and  $\mathbb{T}^n/S_2$  we proceed: Assume that  $\dim \bar{S}_1 = m_1 < m$  and  $\dim (\bar{S}_2/S_2) = m - m_1$  ( $\bar{S}_2/S_2$  is a vector subspace of  $\mathbb{T}^n/S_2$ ). Now, identify  $(\mathbb{T}^n/S_2)/(\bar{S}_2/S_2)$  with any direct summand of  $\bar{S}_2/S_2$  in  $\mathbb{T}^n/S_2$ . Let the  $\bar{T}_1, \bar{T}_2 \in \mathbb{T}^{m \times m_1}$  be chosen such that the columns of  $T_1 \bar{T} \in \mathbb{T}^{n \times m_1}$  span  $\bar{S}_1$  and the columns of  $\bar{T}_2$  span a direct summand of  $(\bar{S}_2/S_2)$  in  $\mathbb{T}^n/S_2 \cong \hat{S}_2$  (with basis furnished by  $T_2$ ). Then, as in Theorem 2.1, the

matrix representation of  $A(\text{mod } S_2)(\text{mod } \bar{S}_2/S_2) \Big|_{\bar{S}_1}$  is given by

$$(\bar{T}_2^* \quad \bar{T}_2)^{-1} \bar{T}_2^* ((T_2^* \quad T_2)^{-1} T_2^* A T_1) \bar{T}_1 \in \mathbb{C}^{m_1 \times m_1}.$$

Eigenvalues of this operator are defined as before, i.e.  $\lambda$  is an eigenvalue of  $A(\text{mod } S_2)(\text{mod } \bar{S}_2/S_2) \Big|_{\bar{S}_1}$  if  $\exists x \neq 0 \in \bar{S}_1$  such that

$$(A - \lambda I)x (\text{mod } S_2)(\text{mod } \bar{S}_2/S_2) = 0$$

Since  $A(\text{mod } \bar{S}_2) \Big|_{\bar{S}_1}$  is well defined in its own right it is natural

to ask for the relation between  $A(\text{mod } S_2)(\text{mod } \bar{S}_2/S_2) \Big|_{\bar{S}_1}$  and

$A(\text{mod } \bar{S}_2) \Big|_{\bar{S}_1}$ . In general, there is none since it is not true that

$$\mathbb{C}^n/\bar{S}_2 \cong (\mathbb{C}^n/S_2)/(\bar{S}_2/S_2). \quad (2.9)$$

If however  $\bar{S}_2 \supset S_2$  then (2.9) is true and  $A(\text{mod } S_2)(\text{mod } \bar{S}_2/S_2) \Big|_{\bar{S}_1}$  and

$A(\text{mod } \bar{S}_2) \Big|_{\bar{S}_1}$  are abstractly\* the same operator. As in the previous

section, we specialize now to the case when  $S_1 \oplus S_2 = \mathbb{C}^n$ . Recall that the term eigenvalue of  $A(\text{mod } S_2) \Big|_{S_1}$  is intuitive under this assumption.

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\* (modulo the isomorphism between  $\mathbb{C}^n/\bar{S}_2$  and  $(\mathbb{C}^n/S_2)/(\bar{S}_2/S_2)$ )

For the eigenvalues of  $A(\text{mod } S_2) (\text{mod } \bar{S}_2/S_2) \Big|_{\bar{S}_1}$  to be intuitive we consider

the case when

$$\tilde{I} \cdot \bar{S}_1 \oplus \bar{S}_2/S_2 = \mathbb{C}^n/S_2 \quad (2.10)$$

Since  $\tilde{I}$  is the canonic isomorphism between  $S_1$  and  $\mathbb{C}^n/S_2$ ; condition (2.10) resembles the condition that  $S_1 \oplus S_2 = \mathbb{C}^n$ . When condition (2.10) holds  $\tilde{S}_1$  and  $(\mathbb{C}^n/S_2)/(\bar{S}_2/S_2)$  are also naturally isomorphic; via

$$\begin{array}{ccccccc} \bar{S}_1 & \xrightarrow{\tilde{i}} & S_1 & \xrightarrow{i} & \mathbb{C}^n & \xrightarrow{I} & \mathbb{C}^n \\ & & & & & & \downarrow P \\ & & & & & & \mathbb{C}^n/S_2 \supset \bar{S}_2/S_2 \\ & & & & & & \downarrow \bar{P} \\ & & & & & & (\mathbb{C}^n/S_2)/(\bar{S}_2/S_2) \end{array}$$

$\tilde{I} = \bar{P} \cdot P \cdot I \cdot i \cdot \tilde{i}$

Definitions 2.6, 2.8; Proposition 2.7 and Theorem 2.8 can now be naturally extended to this setting. It is worthwhile noting that in the case that  $\bar{S}_2 \supset S_2$ , condition (2.10) is equivalent to  $\bar{S}_1 \oplus \bar{S}_2 = \mathbb{C}^n$ .

Further, the process of nesting can be enlarged to  $\dots \bar{\bar{S}}_1 \subset \bar{S}_1 \subset S_1$  in the domain and by successively projecting into  $\mathbb{C}^n/S_2$ ,  $(\mathbb{C}^n/S_2)/(S_2/\bar{S}_2)$ ,  $((\mathbb{C}^n/S_2)/(\bar{S}_2/S_2))/((\bar{S}_2/S_2)/(\bar{S}_2/S_2))$ , ... in the range.

### 2.3 Computation with Orthogonal Projections

A particularly neat computational form appears when we specialize the above definitions and propositions to orthogonal bases and orthogonal projections. Let the subspace  $\hat{S}_2 \subset \mathbb{C}^n$  be the orthogonal complement of  $S_2$  and the columns of  $P_1, P_2 \in \mathbb{C}^{n \times m}$  form orthogonal bases for  $S_1, \hat{S}_2$  respectively. The representation for  $A(\text{mod } S_2) \Big|_{S_1}$  with respect to this

basis is  $P_2^* A P_1 \in \mathbb{C}^{m \times m}$  (by Theorem 2.1). We also denote  $P_2^* A P_1$  by  $A|_{S_1 \rightarrow \hat{S}_2}$ , since it suggests restriction in the domain ( $S_1$ ) and in the range (onto the orthogonal complement of  $S_2$ ).

Restricting attention to the case when  $S_1 \oplus S_2 = \mathbb{C}^n$  (note that  $\hat{S}_2$ , the orthogonal complement of  $S_2$  is different in general from  $S_1$ ); the eigenvalues of  $A(\text{mod } S_2)|_{S_1}$  are the zeros of the polynomial

$$\det(\lambda P_2^* P_1 - P_2^* A P_1) = 0 \quad (2.11)$$

With  $P_1, P_2$  as choice of coordinates a right inverse of  $A(\text{mod } S_2)|_{S_1}$  is given by any right inverse of the matrix  $(P_2^* A P_1) \in \mathbb{C}^{m \times m}$  and is denoted  $A^\dagger|_{\hat{S}_2 \rightarrow S_1}$ . In the computations of section 4, the right inverse of

convenience is the Moore-Penrose inverse. For nested projections; let  $\bar{P}_1 \in \mathbb{C}^{m_1 \times m}$  be an orthogonal basis for  $\bar{S}_1 \subset S_1$  (with  $P_1 \in \mathbb{C}^{m \times n}$  chosen as basis for  $S_1$ ). Further let  $\bar{P}_2 \in \mathbb{C}^{m_1 \times m}$  be an orthogonal basis for the orthogonal direct sum of  $\bar{S}_2/S_2$  in  $\mathbb{C}^n/S_2 \cong \hat{S}_2$  (with  $P_2 \in \mathbb{C}^{n \times m}$  chosen as basis for  $\hat{S}_2$ ). Then, the matrix representation of  $A(\text{mod } S_2)(\text{mod } \bar{S}_2/S_2)|_{S_1}$  is

$$\bar{P}_2^* P_2^* A P_1 \bar{P}_1 \in \mathbb{C}^{m_1 \times m_1}.$$

### Section 3. System Description, Assumptions and Main Formulae

The system under study is the system of Figure 1, where  $G(s)$  is the  $m \times m$  transfer function matrix of a linear, time-invariant, strictly proper control system assumed to have Taylor expansion about  $s = \infty$  (convergent  $\forall |s| > M$ );

$$G(s) = \frac{G_1}{s} + \frac{G_2}{s^2} + \frac{G_3}{s^3} + \dots \quad (3.1)$$

with  $G_1, G_2, \dots \in \mathbb{R}^{m \times m}$ ;  $k$  real and positive. We consider the case when  $G(s)$  is a strictly proper rational transfer function matrix, i.e.  $G(s) \in \mathbb{R}(s)^{m \times m}$  (Formally, all the results of Section 3.1 go through for strictly proper irrational transfer functions with convergent Taylor series at  $s = \infty$ ). We study the closed loop poles of the system of Figure 1 as  $k \rightarrow \infty$ . The motivation is that  $G(s)$  represents the composition of a linear, time-invariant plant and controller and  $k$  represents high gain feedback; as  $k \rightarrow \infty$  the gain tends to  $\infty$  in all control channels. The one-parameter curves traced on an appropriately defined Riemann surface [2] by the closed-loop eigenvalues (parametrized by  $k$ ) are referred to as the multivariable root loci. As  $k \rightarrow \infty$  some of the root loci tend to finite points in (copies of) the complex plane located at the (McMillan) zeros of  $G(s)$  (see for e.g. [3,4,5]), the others go to  $\infty$  as  $k \rightarrow \infty$  and are referred to as the unbounded root loci of the system. We classify the unbounded root loci by the velocity (with  $k$ ) with which they tend to  $\infty$ :

Definition 3.1 An unbounded multivariable root locus  $s_n(k)$  is said to be an nth order unbounded root locus ( $n = 1, 2, 3, \dots$ ) if asymptotically

$$s_n(k) = \mu_n(k)^{1/n} + o(k^{1/n}) \quad (3.2)$$

where  $|\mu_n|$  is a finite constant and  $\lim_{k \rightarrow \infty} o(k^{1/n})/k^{1/n} = 0$

We identify an nth order unbounded root locus with  $\mu_n$ , the coefficient of its asymptotic value.

Theorem 3.2 (Generalized eigenvalue problem for the nth order unbounded root locus)

$0 \neq \mu_n = (-\lambda)^{1/n} \in \mathbb{C}$  is the coefficient of the asymptotic value of an nth order unbounded root locus iff  $\lambda$  is a solution of the generalized eigenvalue problem

$$\begin{bmatrix} G_n - \lambda I & G_{n-1} & \cdot & \cdot & G_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ G_{n-1} & G_{n-2} & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ G_2 & G_1 & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ G_1 & 0 & \cdot & \cdot & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix} = 0 \quad (3.3)$$

where  $v_1 \neq 0, v_2, \dots, v_n \in \mathbb{C}^m$

Proof: Assume, asymptotically, that the value  $s$  of the unbounded root locus is given by

$$s = \mu k^{1/n} + o(k^{1/n}) \quad (3.4)$$

with  $\mu \neq 0$ .

Using the standard method to find the terms in the asymptotic expansion of an implicitly defined variable [6, Chap. 3 esp. article 8] we rewrite  $\det(I + kG(s)) = 0$  asymptotically as:  $\exists e_1 \neq 0, e_2, \dots \in \mathbb{C}^m$  such that

$$\left[ I + \frac{kG_1}{s} + \frac{kG_2}{s^2} \dots \right] \left[ e_1 + \frac{e_2}{k^{1/n}} + \dots + \frac{e_n}{k^{\frac{n-1}{n}}} + \dots \right] = 0 \quad (3.5)$$

or using (3.4), with  $o(k^{1/n})$  neglected

$$\left[ I + \frac{k^{\frac{n-1}{n}} G_1}{\mu} + \frac{k^{\frac{n-2}{n}} G_2}{\mu^2} \dots \right] \left[ e_1 + \frac{e_2}{k^{1/n}} + \dots + \frac{e_n}{k^{\frac{n-1}{n}}} \dots \right] = 0 \quad (3.6)$$

Equating terms of  $o(k^{\frac{n-1}{n}})$ ,  $o(k^{\frac{n-2}{n}})$ ,  $\dots$ ,  $o(1)$  we obtain from (3.6)

$$\begin{bmatrix} G_n + \mu^n I & G_{n-1} & \cdot & \cdot & G_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ G_{n-1} & G_{n-2} & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ G_2 & G_1 & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ G_1 & 0 & \cdot & \cdot & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ \mu e_2 \\ \cdot \\ \cdot \\ \mu^{n-1} e_n \end{bmatrix} = 0 \quad (3.7)$$

(3.3) now follows readily from (3.7), provided of course that (3.3) is non-degenerate (i.e. depends on  $\lambda$ ).  $\square$

Comments: (i) There are  $n$  separate  $n$ th order root loci corresponding to the distinct  $n$ th roots of 1 associated with each solution of (3.3); so that these root loci constitute at  $\infty$ , an  $n$ -cycle of the Riemann surface of the system (see [8, pg. 32], [2, pg. 112]). (ii) The matrix of (3.3) is a triangular, block Toeplitz matrix so that (3.3) must admit of simplification. We take this up next:

### 3.1 Formulae for the asymptotic values of the unbounded root loci

#### 3.1.1 First-Order

Clearly, these are the negatives of the non-zero eigenvalues of  $G_1$

$$s_{i,1} = -\lambda_{i,1} k + o(k), \quad \lambda_{i,1} \in \sigma(G_1), \lambda_{i,1} \neq 0$$

#### 3.1.2 Second Order

These are given by

$$s_{i,2} = (-\lambda_{i,2} k)^{1/2} + o(k^{1/2})$$

where  $\lambda_{i,2}$  is a non zero solution of :  $\exists u_0 \neq 0, u_1$  such that

$$\begin{bmatrix} G_2 - \lambda I & & G_1 \\ & \ddots & \\ & & G_1 & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ \\ u_1 \end{bmatrix} = 0$$

From Proposition 2.3, then  $\lambda_{i,2}$  is a non-zero eigenvalue of

$$G_2(\text{mod } R(G_1)) \Big|_{N(G_1)} \triangleq: \hat{G}_2$$

#### 3.1.3 Third Order

These are given by

$$s_{i,3} = (-\lambda_{i,3} k)^{1/3} + o(k^{1/3})$$

where  $\lambda_{i,3}$  is a nonzero solution of :  $\exists u_0 \neq 0, u_1, u_2$  such that

$$\begin{bmatrix} G_3 - \lambda I & & G_2 & & G_1 \\ & \ddots & & \ddots & \\ & & G_2 & & 0 \\ & & & \ddots & \\ & & & & G_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ \\ u_1 \\ \\ u_2 \end{bmatrix} = 0 \quad (3.8)$$



Proposition 3.3 (Third order eigenvalue formula)

$\lambda_{i,3}$  is a non-zero eigenvalue of

$$(G_3 - G_2 G_1^\dagger G_2)(\text{mod } R(G_1)) \quad (\text{mod } R(\hat{G}_2)) \Big|_{N(\hat{G}_2)}$$

where

$$\hat{G}_2: = G_2(\text{mod } R(G_1)) \Big|_{N(G_1)}$$

and  $G_1^\dagger$  is any right inverse of  $G_1$ .

Remark: Pictorially, we have

$$\begin{array}{c} N(\hat{G}_2) \xrightarrow{i_2} N(G_1) \xrightarrow{i_1} \mathbb{C}^m \xrightarrow{G_3 - G_2 G_1^\dagger G_2} \mathbb{C}^m \\ \downarrow P_1 \\ \mathbb{C}^m / R(G_1) \\ \downarrow P_2 \\ (\mathbb{C}^m / R(G_1)) / R(\hat{G}_2). \end{array}$$

Proof: Let  $v_1, v_2, v_3 \in \mathbb{R}^m$  with  $v_1 \neq 0$  be such that

$$(G_3 - \lambda I)v_1 + G_2 v_2 + G_1 v_3 = 0 \quad (3.9)$$

$$G_2 v_1 + G_1 v_2 = 0 \quad (3.10)$$

$$G_1 v_1 = 0 \quad (3.11)$$

(3.11) yields that  $v_1 \in N(G_1)$ . Next (3.10) yields that

$$v_1 \in N(G_2 (\text{mod } R(G_1)) \Big|_{N(G_1)}) \quad \text{i.e. } v_1 \in N(\hat{G}_2).$$

Further from (3.10), we obtain

$$v_2 = -G_1^\dagger G_2 v_1 + u_1 \quad (3.12)$$

where  $u_1$  is (any) vector belonging to  $N(G_1)$  and  $G_1^\dagger$  is a right-inverse of

$G_1$ . Using (3.12) in (3.9) we obtain

$$(G_3 - G_2 G_1^\dagger G_2 - \lambda I)v_1 + G_2 u_1 + G_1 v_3 = 0.$$

$$\text{i.e. } (G_3 - G_2 G_1^\dagger G_2 - \lambda I)v_1 \bmod R(G_1) \bmod R(\hat{G}_2) = 0.$$

This proves the proposition.

### 3.1.4 Fourth Order

These are of the form

$$s_{i,4} = (-\lambda_{i,4} k)^{1/4} + o(k^{1/4})$$

#### Proposition 3.4 (Fourth Order Eigenlocus Formula)

$\lambda_{i,4}$  is a non zero eigenvalue of

$$\begin{aligned} & (G_4 - G_3 G_1^\dagger G_2 - G_2 G_1^\dagger G_3 + G_2 G_1^\dagger G_2 G_1^\dagger G_2 - G_3 \hat{G}_2^\dagger \tilde{G}_3 + \\ & + G_2 G_1^\dagger G_2 \hat{G}_2^\dagger \tilde{G}_3) \bmod R(G_1) \bmod R(\hat{G}_2) \bmod R(\hat{G}_3) \Big|_{N(\hat{G}_3)} \end{aligned}$$

$$\text{where } \tilde{G}_3 = (G_3 - G_2 G_1^\dagger G_2) \bmod R(G_1) \Big|_{N(G_1)},$$

$$\hat{G}_3 = (G_3 - G_2 G_1^\dagger G_2) \bmod R(G_1) \bmod R(\hat{G}_2) \Big|_{N(\hat{G}_2)},$$

$$G_1^\dagger \text{ is a right inverse of } G_1$$

$$\text{and } \hat{G}_2^\dagger \text{ is a right inverse of } \hat{G}_2.$$

Proof: From Theorem 3.2,  $\lambda_{i,4}$  is the coefficient of a fourth order root locus if  $\exists v_1 \neq 0, v_2, v_3, v_4 \in \mathbb{C}^m$  such that

$$(G_4 - \lambda I)v_1 + G_3 v_2 + G_2 v_3 + G_1 v_4 = 0 \quad (3.13)$$

$$G_3 v_1 + G_2 v_2 + G_1 v_3 = 0 \quad (3.14)$$

$$G_2 v_1 + G_1 v_2 = 0 \quad (3.15)$$

$$G_1 v_1 = 0 \quad (3.16)$$

As before from (3.15), (3.16) we have  $v_1 \in N(\hat{G}_2) \subset N(G_1)$  and  $v_2 = -G_1^\dagger G_2 v_1 + u_1$  for some  $u_1 \in N(G_1)$ . Using this in (3.14) we get

$$(G_3 - G_2 G_1^\dagger G_2) v_1 + G_2 u_1 + G_1 v_3 = 0 \quad (3.17)$$

Thus  $v_1 \in N(\hat{G}_3)$  and

$$v_3 = -G_1^\dagger (G_3 - G_2 G_1^\dagger G_2) v_1 - G_1^\dagger G_2 u_1 + u_2 \quad (3.18)$$

for some  $u_2 \in N(G_1)$ .

Further from (3.17) we have

$$((G_3 - G_2 G_1^\dagger G_2) v_1 + G_2 u_1) \pmod{R(G_1)} = 0 \quad (3.19)$$

Since  $u_1 \in N(G_1)$  we may solve (3.19) to obtain

$$u_1 = -\hat{G}_2^\dagger ((G_3 - G_2 G_1^\dagger G_2) v_1 \pmod{R(G_1)}) + \bar{u}_1 \quad (3.20)$$

with  $\bar{u}_1 \in N(\hat{G}_2)$ . (3.20) is written in the notation of this proposition as

$$u_1 = -\hat{G}_2^\dagger \tilde{G}_3 v_1 + \bar{u}_1. \quad (3.21)$$

Using (3.21) and (3.18) in (3.13) we obtain

$$\begin{aligned} & (G_4 - G_3 G_1^\dagger G_2 - G_2 G_1^\dagger G_3 + G_2 G_1^\dagger G_2 G_1^\dagger G_2 - G_3 \hat{G}_2 \tilde{G}_3 + G_2 G_1^\dagger G_2 \hat{G}_2^\dagger \tilde{G}_3 - \lambda I) v_1 \\ & + (G_3 - G_2 G_1^\dagger G_2) \bar{u}_1 + G_2 u_2 + G_1 v_4 = 0 \end{aligned} \quad (3.22)$$

with  $\bar{u}_1 \in N(\hat{G}_2)$ ,  $u_2 \in N(G_1)$ ,  $v_4$  arbitrary. From (3.22) the statement of the proposition is obvious. □

### 3.1.5 Higher Order

The general formulae are messy but are based on straightforward considerations: The basic idea is to solve the triangular set of equations

as if  $G_1, \hat{G}_2, \hat{G}_3, \dots$  were invertible using right inverses  $G_1^+, \hat{G}_2^+, \hat{G}_3^+$ .

The non-uniqueness in this process is kept track off by successively restricting in the domain to  $N(G_1), N(\hat{G}_2), \dots$ . Also, the conditions for the algebraic equations to have solutions are kept track of by succesively modding out in the range  $R(G_1), R(\hat{G}_2), \dots$ .

### 3.2 Simple Null Structure and Integer Order for All Unbounded Root Loci

In general, the branches of the algebraic function obtained from  $\det(I + kG(s)) = 0$  at  $s =$  have asymptotic expansion (see [8])

$$s = \lambda(k)^{m/n} + o(k^{m/n})$$

showing possible non-integral order, unbounded root loci.

However, we show now that under some simple assumptions the only unbounded root loci are those of integral order. First, some preliminaries

Notation: Define  $T_n \in \mathbb{R}^{nm \times nm}$  to be the block Toeplitz matrix:

$$T_n = \begin{pmatrix} G_n & G_{n-1} & \dots & \dots & G_1 \\ G_{n-1} & \dots & \dots & \dots & 0 \\ \vdots & \dots & \dots & \dots & \vdots \\ G_1 & 0 & \dots & \dots & 0 \end{pmatrix}$$

Assumption 1 (Non-degeracy assumption)

There exists some  $n_0$  such that

$$R \begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix} \subset R(T_n) \quad (3.23)$$

where the matrix on the left hand side of (3.23) has  $n_0 - 1$  blocks of  $m \times m$  zero matrices.

Comments: (i) If (3.23) is satisfied for some  $n_0$ , then for all  $n \geq n_0$

$$R \begin{pmatrix} I \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \subset R(T_n)$$

with the matrix on the left hand side having  $n-1$  blocks of zero matrices. Hence, in the sequel we will understand that  $n_0$  is the smallest integer such that an equation of the form (3.23) holds.

(ii) (3.23) implies in particular that no linear combination of outputs is identically zero. Further insight into the nature of this assumption follows from Proposition 3.5.

To study the behavior at  $s = \infty$  of  $G(s)$  perform the change of variables,  $w = 1/s$ . Then,

$$G(w) = G_1 w + G_2 w^2 + \dots$$

Recall that  $G(w)$  admits of a unique Smith-McMillan form  $\Lambda(w)$  given by

$$G(w) = M(w) \Lambda(w) N(w)$$

where  $M(w)$  and  $N(w)$  are unimodular matrices and

$$\Lambda(w) = \text{diag} \left( \frac{e_1(w)}{f_1(w)}, \dots, \frac{e_m(w)}{f_m(w)} \right) \quad .$$

with the  $e_i$  and  $f_i$  monic coprime polynomials, with  $e_i$  dividing  $e_{i+1}$  and  $f_{i+1}$  dividing  $f_i$  for all  $i$ . Further

Proposition 3.5 (Explication of Assumption 1)

Assumption 1  $\Leftrightarrow G(w)$  has normal rank  $m$ .

□

Proof: Follows readily from the techniques of [7].

For any  $\alpha \in \mathbb{C}$ , which is either a pole or zero of  $G(w)$

$$\Lambda(w) = \Lambda_{\alpha}(w) \cdot \tilde{\Lambda}(w)$$

where

$$\Lambda_{\alpha} = \begin{bmatrix} (w-\alpha)^{\sigma_1} & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & \ddots & \\ & & 0 & & \ddots \\ & & & & & (w-\alpha)^{\sigma_m} \end{bmatrix} \quad (3.24)$$

with  $\sigma_1 \leq \sigma_2 \leq \dots \leq \sigma_m$ .

The matrix  $\Lambda_{\alpha}(w)$  contains information about the order  $\omega_p(\omega_z)$  and degree  $\delta_p(\delta_z)$  of the pole (zero) at  $w = \alpha$  as follows

$$\omega_p = -\sigma_1 \text{ if } \sigma_1 \leq 0 \quad \delta_p = -\sum_{\sigma_i < 0} \sigma_i \quad (3.25)$$

$$\omega_z = \sigma_m \text{ if } \sigma_m \geq 0 \quad \delta_z = \sum_{\sigma_i > 0} \sigma_i \quad (3.26)$$

We are interested in the order and degree of the zero of  $G(w)$  at  $w = 0$ . A theorem of [7] relates  $\omega_z, \delta_z$  for the zero at  $w = 0$  to the ranks of the block Toeplitz matrices  $T_n$ , defined by (3.19). Define, for  $i \geq 1$  (with the understanding that  $T_0 = 0$ )

$$\rho_i = \text{rank } T_i - \text{rank } T_{i-1} \quad .$$

Then, we have

Theorem 3.6 [7] (Order and degree of zeros related to rank  $T_n$ )

$$\omega_z = \min \{i | \rho_i = m\} \quad (3.27)$$

$$\delta_z = \sum_{i=1}^{\omega_z} (m - \rho_i) + m \quad (3.28)$$

Corollary 3.7 (Further explication of Assumption 1) [7].

Let  $n_0$  be the smallest integer so that (3.23) is satisfied. Then,

$$n_0 = \omega_z.$$

Comment: The number of unbounded root loci of the system is  $\delta_z$ . We will show under Assumption 2, that  $\delta_z$  unbounded roots of the 1st, 2nd, ...,  $\omega_z$ th order are obtained. First, a preliminary proposition.

Proposition 3.8 (Connection between rank  $\hat{G}_n$  and rank  $T_n$ )

$$\text{rank } G_1 = \rho_1$$

$$\text{rank } \hat{G}_2 := \dim (R(\hat{G}_2)) = \rho_2 - \rho_1$$

$$\text{rank } \hat{G}_3 := \dim (R(\hat{G}_3)) = \rho_3 - \rho_2$$

and so on.

Proof: The proposition follows from the observation that:

$$\text{rank } T_1 = \text{rank } G_1 = \rho_1$$

$$\text{rank } T_2 - \text{rank } T_1 = \text{rank } G_1 + \text{rank } \hat{G}_2 = \rho_2$$

$$\text{rank } T_3 - \text{rank } T_2 = \text{rank } G_1 + \text{rank } \hat{G}_2 + \text{rank } \hat{G}_3 = \rho_3$$

and so on. □

Recall from Proposition 2.7 that the connection between rank and number of non-zero eigenvalues is simple-null structure. Hence, we assume

Assumption 2 (Simple null structure)

Assume that

$$G_1$$

$$\hat{G}_2 := G_2 \bmod R(G_1) \Big|_{N(G_1)}$$

$$\hat{G}_3 := (G_3 - G_2 G_1^+ G_2) \bmod R(\hat{G}_2) \bmod R(G_1) \Big|_{N(\hat{G}_2)}$$

etc. have simple null structure.

The key observation to make is that Assumption 2 guarantees that the operators  $G_1, \hat{G}_2, \hat{G}_3, \dots$ , are of the form studied in the previous section (i.e. of the form  $A(\text{mod } S_2) \big|_{S_1}$  with  $S_1 \oplus S_2 = \mathbb{C}^m$ ), and nested restrictions of the same kind. Precisely,

Proposition 3.9

$G_1$  has simple null structure  $\Leftrightarrow R(G_1) \oplus N(G_1) = \mathbb{R}^m$ .

Proof: Follows from Jordan canonical form. □

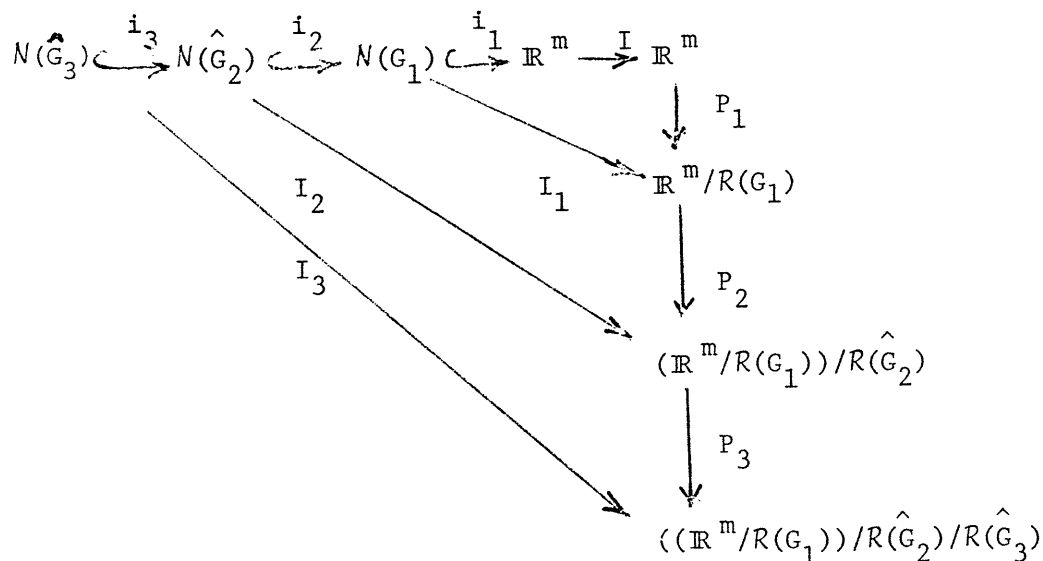
Let  $I_1$  now be the (natural) isomorphism between  $N(G_1)$  and  $\mathbb{R}^m/R(G_1)$ . Then, we have

Proposition 3.10

$\hat{G}_2$  has simple null structure  $\Leftrightarrow I_1 N(\hat{G}_2) + R(\hat{G}_2) = \mathbb{R}^m/R(G_1)$ .

Proof: exactly as in Proposition 3.9 □

Similar considerations hold for  $\hat{G}_3, \hat{G}_4$ , etc.. Pictorially, we have the condition that the operators  $I_1, I_2, I_3, \dots$ , etc. defined so that the diagram below commutes are all isomorphisms.



Proposition (2.7) then assures us that the number



of non-zero eigenvalues of  $G_1, \hat{G}_2, \hat{G}_3, \dots$  is the same as  $\text{rank } G_1, \text{rank } \hat{G}_2, \text{rank } \hat{G}_3, \dots$ . Using this, we obtain

Theorem 3.11 (Asymptotic Unbounded Root Loci)

Under Assumptions 1 and 2 the only unbounded root loci of the system of Figure 1 are the 1st, 2nd, ...,  $n_0$ th order unbounded root loci specified by Theorem 3.2.

Proof: The proof is by counting. By Theorem 3.2 and the observations made above the number of 1st, 2nd, ...,  $n_0$ th order unbounded root loci is

$$\dim R(G_1) + 2 \dim R(\hat{G}_2) + \dots + n_0 \dim R(\hat{G}_{n_0})$$

Using Proposition 3.8, this is rewritten as

$$\rho_1 + 2(\rho_2 - \rho_1) + \dots + n_0(\rho_{n_0} - \rho_{n_0-1}) \quad (3.29)$$

with  $\rho_{n_0} = m$  and  $n_0 = \omega_z$  (by Theorem 3.6 and Corollary 3.7).

Simplifying (3.29) we obtain the number of 1st, ...,  $n_0$ th order unbounded root loci to be

$$m + \sum_{i=1}^{n_0} (m - \rho_i) = \delta_z.$$

This proves the theorem. □

Comments on dropping the simple null structure assumption

Consider the following example of the failure to Theorem 3.11 without the simple null structure assumption:

$$G_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad G_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad G_k = 0 \quad k \geq 3.$$

By Theorem 3.2 there are no first, second, .... order unbounded root loci.

However there are unbounded root loci.

Indeed by Theorem 3.6  $\omega_z = 2$  and  $\delta_z = 3$ . Note that  $\det(I+kG(s)) = 0$

yields  $1 + \frac{k^2}{s^3} = 0$  so that there are three  $O(k^{2/3})$  root loci.

The basic point is this - in the absence of the simple null structure assumption, the unbounded root loci may be computed by examining the perturbation of the nilpotent blocks, which violate the assumption. The main tool is the following result in numerical analysis (see [9]):

The eigenvalues of

$$\begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix} + B(\epsilon) ,$$

where  $B(\epsilon)$  is an  $m \times m$  matrix of  $O(\epsilon)$  with  $\lim_{\epsilon \rightarrow 0} b_{ml}(\epsilon) \neq 0$  are  $O(\epsilon^{1/m})$  with

leading term in their asymptotic expansion given by  $\epsilon^{1/m} \left[ \lim_{\epsilon \rightarrow 0} b_{ml}(\epsilon) \right]^{1/m}$ .

Thus, for instance, let  $G_1, \dots, G_{i-1} = 0$  and

$$G_i = \begin{bmatrix} 0 & 1 & & \ddots \\ & \ddots & \ddots & \vdots \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}$$

with  $g_{ml}^{(i+1)}$ , the  $(m, l)$ th element of  $G_{i+1}$  not equal to zero. Then there are  $(im+1)$  unbounded root loci of  $O(k^{m/im+1})$  with coefficient of the leading term in their asymptotic series given by  $\left( (-1)^{m+1} g_{ml}^{(i+1)} \right)^{\frac{1}{im+1}}$ .

#### Section 4. Computation of Asymptotic values

The development of section 3 was abstract and the formulae for the coefficients of the asymptotic values were given as the eigenvalues of some abstract maps

$G_1, \hat{G}_2, \hat{G}_3, \dots$ . For computing these eigenvalues in coordinates and to obtain additional geometric insight into the structure of these asymptotic values we use orthogonal projections (as in section 2.3) and the singular value decomposition (see for e.g. [1], [17]).

Proposition 4.1 (The Singular Value Decomposition s.v.d.) [17].

A matrix  $A \in \mathbb{C}^{m \times m}$  of rank  $r$  may be decomposed as

$$A = [U_1 \ ; \ U_2] \begin{bmatrix} \sum_1 & & 0 \\ & \ddots & \\ 0 & & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ - \\ V_2^* \end{bmatrix} \quad (4.2)$$

where  $U = [U_1 \ ; \ U_2] \in \mathbb{C}^{m \times m}$  with  $U_1 \in \mathbb{C}^{m \times r}$ ;  $U_2 \in \mathbb{C}^{m \times (m-r)}$

and  $V = [V_1 \ ; \ V_2] \in \mathbb{C}^{m \times m}$  with  $V_1 \in \mathbb{C}^{m \times r}$ ;  $V_2 \in \mathbb{C}^{m \times (m-r)}$

are unitary matrices and  $\sum_1 \in \mathbb{R}_+^{r \times r}$  is a diagonal matrix of positive real numbers.

Comment: The columns of  $V_1, U_1$  represent orthogonal bases for the range spaces of  $A^*, A$  respectively. The columns of  $V_2, U_2$  represent orthogonal bases for the null spaces of  $A, A^*$  respectively.

Notation: Denote the s.v.d. of  $G_1 \in \mathbb{R}^{m \times m}$  by

$$G_1 = [U_1^1 \ ; \ U_2^1] \begin{bmatrix} \sum_1^1 & & 0 \\ & \ddots & \\ 0 & & 0 \end{bmatrix} \begin{bmatrix} V_1^{1*} \\ - \\ V_2^{1*} \end{bmatrix} \quad (4.2)$$

where  $\sum_1^1 \in \mathbb{R}_+^{m_1 \times m_1}$  and the other matrices are real and of conformal dimensions.

Further let the s.v.d. of  $U_2^{1*} G_2 V_2^1 \in \mathbb{R}^{(m-m_1) \times (m-m_1)}$  be given by

$$U_2^{1*} G_2 V_2^1 = [U_1^2 \ ; \ U_2^2] \begin{bmatrix} \sum_1^2 & & 0 \\ & \ddots & \\ 0 & & 0 \end{bmatrix} \begin{bmatrix} V_1^{2*} \\ - \\ V_2^{2*} \end{bmatrix} \quad (4.3)$$

where  $\sum_1^2 \in \mathbb{R}_+^{m_2 \times m_2}$ .

Denote the S.V.D. of  $U_2^{2*} U_2^{1*} (G_3 - G_2 G_1^\dagger G_2) V_2^1 V_2^2 \in \mathbb{R}^{(m-m_1-m_2) \times (m-m_1-m_2)}$  by

$$U_2^{2*} U_2^{1*} (G_3 - G_2 G_1^\dagger G_2) V_2^1 V_2^2 = [U_1^3 \quad ; \quad U_2^3] \begin{bmatrix} \sum_1^3 & \vdots & 0 \\ - & - & - \\ 0 & \vdots & 0 \end{bmatrix} \begin{bmatrix} V_1^{3*} \\ - & - & - \\ V_2^{3*} \end{bmatrix} \quad (4.4)$$

choosing for  $G_1^\dagger$ , the Moore-Penrose inverse,

$$[V_1^1 \quad V_2^1] \begin{bmatrix} 1 & & \\ (\sum_1)^{-1} & \vdots & 0 \\ - & - & - \\ 0 & \vdots & 0 \end{bmatrix} \begin{bmatrix} U_1^{1*} \\ - & - & - \\ U_2^{1*} \end{bmatrix} .$$

Then, using (2.11) to compute the asymptotes of the integral order asymptotic root loci computed in Section 3.1 we obtain

Theorem 4.2 (Polynomial equations for the asymptotes of integral order unbounded root loci).

- (i)  $\mu_1 = -\lambda \neq 0$  is the coefficient of the asymptotic value of a 1st order unbounded root locus iff

$$\det (-G_1 + \lambda I) = 0 \quad (4.5)$$

- (ii)  $\mu_2 = -\lambda^{1/2} \neq 0$  is the coefficient of the asymptotic value of a 2nd order unbounded root locus iff

$$\det (\lambda U_2^{1*} V_2^1 - U_2^{1*} G_2 V_2^1) = 0 \quad (4.6)$$

- (iii)  $\mu_3 = -\lambda^{1/3} \neq 0$  is the coefficient of the asymptotic value of a 3rd order unbounded root locus iff

$$\det (\lambda U_2^{2*} U_2^{1*} V_2^1 V_2^2 - U_2^{2*} U_2^{1*} (G_3 - G_2 G_1^\dagger G_2) V_2^1 V_2^2) \neq 0, \quad (4.7)$$

and so on.

Comments: (i) Note that equations (4.5), (4.6), (4.7) are solved by setting up generalized eigenvalue problems of dimension  $m, m-m_1, m-m_1-m_2$  respectively. They can in fact be set up as ordinary eigenvalue problems, since by Assumption 2,  $U_2^{1*} V_2^1, U_2^{2*} V_2^1, U_2^{1*} V_2^2, \dots$  are invertible.

(ii) Some geometrical insight into the computation procedures is obtained by expressing the solutions of (4.5), (4.6), (4.7) as the non-zero eigenvalues of

$$\begin{aligned} G_1 & \\ G_2 \Big|_{N(G_1) \rightarrow N(G_1^*)} & =: \hat{G}_2 \quad ; \\ G_3 - G_2 G_1^\dagger G_2 \Big|_{N(\hat{G}_2) \rightarrow N(\hat{G}_2^*)} & =: \hat{G}_3 \quad ; \quad \text{respectively.} \end{aligned}$$

Theorem (4.2) then identifies orthogonal subspaces of the input space ( $\mathbb{R}^m$ ) and the output space ( $\mathbb{R}^m$ ) for the computation of the integral order unbounded root loci.  $R(V_1^1), R(V_2^1 V_1^2), R(V_2^1 V_2^2 V_1^3), \dots$  are subspaces of the input space and  $R(U_1^1), R(U_2^1 U_1^2), R(U_2^1 U_2^2 U_1^3), \dots$  are subspaces of the output space associated with the 1st, 2nd, 3rd, ... order unbounded root loci respectively.

#### Section 5: Pivots for the asymptotic root loci

For the integral order root loci the asymptotic series have the form given by (5.1) below, provided the  $\lambda_i$ 's have simple structure (see[8])

$$s_{i,n} = (-k\lambda_i)^{1/n} + c_i + o(1) + \dots \quad n = 1, 2, 3, \dots, n_0 \quad (5.1)$$

with  $\lambda_i \neq 0, c_i \in \mathbb{C}$ . By the pivot of the asymptotic root locus is meant the coefficient of the  $o(1)$  term of the asymptotic expansion of (5.1) i.e.  $c_i$ . Each cycle of the multivariable root locus at  $\infty$  has the same pivot. To make the calculation we need

Assumption 3 (Simple Structure)

Assume that  $G_1, G_2, G_3, \dots, G_{n_0}$  have simple structure associated with each of their eigenvalues.

Theorem 5.1 (Expression for the pivots)

Under assumptions 1.2,3 the  $n$ th order asymptotic unbounded root loci for the system of Figure 1 have the form (5.1) with  $c_i$  given by the solution of:

$e_1 \neq 0, e_2, \dots, e_{n+1} \in \mathbb{C}^m$  such that

$$\begin{bmatrix} G_{n+1} - c_i G_n & G_n - \lambda_i I & & & G_1 \\ G_n - \lambda_i I & G_{n-1} & & & G_1 \\ & & & & \\ & & & & \\ G_2 & G_1 & & 0 & 0 \\ G_1 & 0 & & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_{n+1} \end{bmatrix} = 0 \quad (5.2)$$

Proof: Using the same technique as in the proof of Theorem 3.2 we get,

$$\text{with } \frac{k}{s^n} = \frac{1}{\mu^n} \left( 1 - \frac{nc}{\mu k^{1/n}} + o(k^{-2/n}) \right)$$

$$\left[ I + \frac{n-1}{k^n \mu} G_1 + \frac{n-2}{k^n \mu^2} G_2 + \dots + \frac{G_n}{\mu^n} - \frac{nG_n c}{\mu^{n+1} k^{1/n}} + \frac{G_{n+1}}{k^{1/n} \mu^{n+1}} + \dots \right] \begin{bmatrix} e_1 + \frac{e_2}{k^{1/n}} + \dots + \frac{e_n}{k^n} + \frac{e_{n+1}}{k} + \dots \end{bmatrix} = 0. \quad (5.3)$$

Equating terms of  $o(k^{-n})$ ,  $\dots$ ,  $o(1)$ ,  $o(k^{-n})$ , one obtains equation (5.2)

with  $-\mu^n = \lambda_{i,n}$ , with  $e_1 \neq 0, e_2, \dots, e_{n+1} \in \mathbb{C}^m$ . □

Comments: (i) For each of the  $n$ th root of  $\lambda_{i,n}$  the same value of  $c_i$  occurs from equation (5.2), justifying the term "pivot of the  $n$ -cycle for  $c_i$ ."

(ii) Equation (5.2) is triangular block Toeplitz and so admits of simplification. We take this up next.

### 5.1 Formulae for the pivots of the unbounded root loci

5.1.1 First Order:  $s = -\lambda_i k + c_i + o(1)$ .

$c_i$  are the solutions of

$$\begin{bmatrix} G_2 - c_i G_1 & \vdots & G_1 - \lambda_i I \\ \hline G_1 - \lambda_i I & \vdots & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0 \quad (5.4)$$

where  $e_1 \neq 0$ ,  $e_2 \in \mathbb{C}^m$  and  $\lambda_i$  is a non-zero eigenvalue of  $G_1$ . This may be rewritten after row operations as

$$\begin{bmatrix} G_2 - c_i \lambda_i I & \vdots & G_1 - \lambda_i I \\ \hline & G_1 - \lambda_i I & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that by Proposition 2.3,  $c_i \lambda_i$  is an eigenvalue of

$$G_2 \bmod R(G_1 - \lambda_i I) \Big|_{N(G_1 - \lambda_i I)} \quad (5.5)$$

By Assumption 3, in analogy to Proposition (3.9) we have

$$R(G_1 - \lambda_i I) \oplus N(G_1 - \lambda_i I) = \mathbb{C}^m$$

so that there are as many eigenvalues to the operator in (5.5) as the dimension of  $N(G_1 - \lambda_i I)$ .

5.1.2 Second Order:  $s = \sqrt{-\lambda_i k} + c_i + o(1)$ .

$c_i$  are the solutions of:  $\exists e_1 \neq 0, e_2, e_3 \in \mathbb{C}^m$  such that

$$\begin{bmatrix} G_3 - 2c_i G_2 & G_2 - \lambda_i I & G_1 \\ \hline G_2 - \lambda_i I & G_1 & 0 \\ \hline G_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 0 \quad (5.6)$$

Proposition 5.1 (Formula for second order pivots)

The second order pivots  $c_i$  corresponding to  $\lambda_i$  of (5.1), the coefficient of a second order unbounded root locus are  $\frac{1}{2\lambda_i}$  times the eigenvalues of

$$(G_3 - (G_2 - \lambda_i I) G_1^\dagger (G_2 - \lambda_i I)) \bmod (R(\hat{G}_2 - \lambda_i \hat{I})) \bmod R(G_1) \Big|_{N(\hat{G}_2 - \lambda_i \hat{I})} \quad (5.7)$$

where  $(\hat{G}_2 - \lambda_i \hat{I}) := G_2 - \lambda_i I \bmod R(G_1) \Big|_{N(G_1)}$ .

Proof: (5.6) may be rewritten as:  $\exists v_1 \neq 0, v_2, v_3 \in \mathbb{C}^m$  such that

$$\begin{bmatrix} G_3 - 2c_i \lambda_i I - 2c_i (G_2 - \lambda_i I) & G_2 - \lambda_i I & G_1 \\ \hline G_2 - \lambda_i I & G_1 & 0 \\ \hline G_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \quad (5.8)$$

Adding  $2c_i$  times the second row of (5.8) to the first row of (5.8) and then subtracting  $2c_i$  times the third column of (5.8) from the second column of (5.8) we have

$$\begin{bmatrix} G_3 - 2c_i \lambda_i I & G_2 - \lambda_i I & G_1 \\ \hline G_2 - \lambda_i I & G_1 & 0 \\ \hline G_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \quad (5.9)$$

(5.7) now follows readily from (5.9).  $\square$



Comment: As before by Assumption 3, there are as many pivots as there are  $\lambda_i$ 's for the second order unbounded root loci.

5.1.3. Third order  $s = (-\lambda_i k)^{1/3} + c_i + o(1)$

$c_i$  are the solutions of  $\exists e_1 \neq 0, e_2, e_3, e_4 \in \mathbb{C}^m$  with

$$\begin{bmatrix} G_4 - 3c_i G_3 & G_3 - \lambda_i I & G_2 & G_1 \\ \hline G_3 - \lambda_i I & G_2 & G_1 & 0 \\ \hline G_2 & G_1 & 0 & 0 \\ \hline G_1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = 0 \quad (5.10)$$

Proposition 5.2 (Formula for third order pivots)

The third order pivots  $c_i$  corresponding to  $\lambda_i$  of (5.1), the coefficient of a third order unbounded root locus are  $\frac{1}{3\lambda_i}$  times the eigenvalues of

$$\begin{aligned} & G_4 - (G_3 - \lambda_i I) G_1^+ G_2 - G_2 G_1^+ (G_3 - \lambda_i I) + G_2 G_1^+ G_2 G_1^+ G_2 - (G_3 - \lambda_i I) \hat{G}_2^+ (\tilde{G}_3 - \lambda_i \tilde{I}) \\ & + G_2 G_1^+ G_2 \hat{G}_2^+ (\tilde{G}_3 - \lambda_i \tilde{I}) \text{ mod } R(G_1) \text{ mod } R(\hat{G}_2) \text{ mod } R(\hat{G}_3 - \lambda_i \hat{I}) \Big| N(\hat{G}_3 - \lambda_i \hat{I}) \end{aligned} \quad (5.11)$$

where  $\hat{G}_3 - \lambda_i \hat{I} := (G_3 - G_2 G_1^+ G_2 - \lambda_i I) \text{ mod } R(G_1) \text{ mod } R(\hat{G}_2) \Big| N(G_2)$  and  $G_3 - \lambda_i I := (G_3 - G_2 G_1^+ G_2 - \lambda_i I) \text{ mod } R(G_1) \Big| N(G_1)$

Proof: Exactly as in Proposition 5.1. □

5.1.4 Higher order pivots  $s = (-\lambda_i k)^{1/n} + c_i + o(1)$

The extension of the foregoing procedure to higher order pivots is exactly as in Section 3.1 and is omitted.

## 5.2 Computation of the Pivots

The machinery introduced in Section 4 can be used to set up a procedure

for the computation of the pivots involving essentially multiplying  $G_2$ ,

$G_3 - (G_2 - \lambda_i I)G_1^\dagger (G_2 - \lambda_i I), \dots$  by  $U_2^{1*}, U_2^{2*} U_2^{1*}, \dots$  on the left and  $V_2^1, V_2^1 V_2^2, \dots$  on the right. The details are omitted.

## Section 6. Concluding Remarks

The above calculations of the asymptotes of the unbounded root loci may be applied to state a necessary and sufficient condition for the closed loop exponential stability of a strictly proper linear time-invariant system under arbitrarily high gain feedback  $k \geq k_0$  as follows:

### Theorem 6.1 (High gain stability)

If the strictly proper, linear time-invariant plant  $G(s)$  satisfies Assumptions 1,2,3; then the closed loop system of Figure 1 is exponentially stable for all  $k \geq k_0$  with all closed loop eigenvalues uniformly (for  $k \in [k_0, \infty]$ ) bounded away from the  $j\omega$ -axis for  $k \geq k_0$ , iff

- (i) the McMillan zeros of  $G(s)$  are in the  $\mathbb{C}_-^\bullet$ ,
- (ii) the non zero eigenvalues of  $G_1$  are in  $\mathbb{C}_+$ ,
- (iii) the eigenvalues of  $G_2(\text{mod } R(G_1)) \Big|_{N(G_1)}$  are real and positive,
- (iv) the pivot associated with each non-zero eigenvalue of  $G_1$  on the  $j\omega$  axis and with each eigenvalue of  $G_2 \text{ mod } R(G_1) \Big|_{N(G_1)}$  has negative real part,
- (v)  $\mathbb{R}^m = R(G_1) + R(G_2 \Big|_{N(G_1)})$ .

Comments: (i) Condition (v) of the theorem guarantees that only first and second order unbounded root loci exist.

(ii) Theorem 6.1 is the generalization to multi input - multi output of a well known theorem for single-input, single output systems (see [13]).

The results of this paper are easily generalized to the case of proper rather than strictly proper plants. The Taylor series about  $s = \infty$  then is

$$G(s) = G_0 + \frac{G_1}{s} + \dots$$

and the calculations would begin with restriction in domain to  $N(G_0)$  and  $\text{mod}R(G_0)$  in the range.

(iv) We have performed formal asymptotic calculations to find the asymptotic expansions for the unbounded root loci in terms of the coefficient matrices of the Laurent expansion of  $G(s)$  about  $s = \infty$  (Markov parameters). It is clear that the same asymptotic calculation for the bounded root loci, i.e. the ones that tend to the McMillan zeros, can be performed in terms of the coefficient matrices of the Laurent expansion of  $G(s)$  about  $s=z_i$  (where  $z_i$  is a McMillan zero of  $G(s)$ ).

We believe the generalization to the case of proper irrational transfer functions analytic outside a compact disc is also immediate. Note, however, that there is no counterpart of the Smith McMillan theory of Section 3.2.

We have not investigated in our set-up the specialization of our computations to asymptotic Linear Quadratic regulators (see for e.g. [14], [15], [16]).

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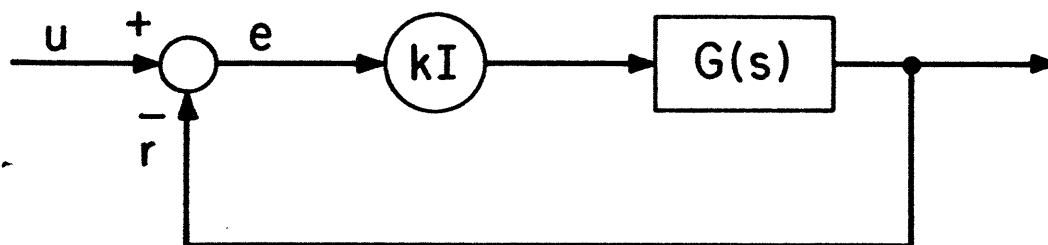


Figure 1. System Configuration